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Bäcklund transformations for the $sl(2)$ Gaudin magnet

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Abstract

Elementary, one- and two-point, Bäcklund transformations are constructed for the generic case of the $sl(2)$ Gaudin magnet. The spectrality property is used to construct these explicitly given, Poisson integrable maps which are time discretizations of the continuous flows with any Hamiltonian from the spectral curve of the 2×2 Lax matrix.

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1. Introduction

Bäcklund transformations (BTs) are an essential tool used to generate new solutions out of given solutions to integrable equations. This is by now a well-developed area, with elegant BTs having been found and studied for almost all integrable hierarchies, see [1, 2].

The theory of BTs for evolution equations entered the subject of finite-dimensional integrability through the discretization of time variable(s). One of the most important and earliest accounts on this subject is in the papers by Veselov [3] where integrable Lagrange correspondences were introduced as discrete-time analogues of integrable continuous flows. Veselov clarified the geometric meaning of these correspondences as finite shifts on Jacobians and gave several important examples. The reader is referred to an extensive literature which has appeared since then: see, for instance, [4–7] and references therein.

In this paper, following the approach of [8], we look at BTs for finite-dimensional (Liouville) integrable systems as special canonical transformations, thereby taking a Hamiltonian point of view. We introduce and study several new properties of BTs which appear to be very natural in such an approach.

BTs for finite-dimensional integrable systems are defined in this paper as symplectic, or more generally Poisson, integrable maps which are explicit maps (rather than the implicit multi-valued correspondences of [3]) and which can be viewed as time discretizations of particular continuous flows. The most characteristic features of such maps are: (i) BTs preserve the same set of integrals of motion as does the continuous flow which they discretize, (ii) they depend on a (Bäcklund) parameter λ that specifies the corresponding shift on a Jacobian or on a generalized Jacobian [9] and (iii) a spectrality property holds with respect to λ and to the 'conjugate' variable μ, which means that the point (λ, μ) belongs to the spectral curve [8, 10].

Because of the above properties, the constructed BTs are suitable as explicit (symplectic) geometric integrators. Explicitness makes these maps purely iterative, while the importance of the parameter λ is that it allows an adjustable discrete time step. The spectrality property is strongly related to the symplecticness of the map. Finally, numerical integrators which exactly preserve the level set of integrals and at the same time are symplectic proved to be impossible to find for generic Hamiltonian dynamics [11], but for integrable flows they do exist and so are in demand.

In this paper we consider a generic (diagonal) case of the $sl(2)$ *XXX* Gaudin magnet which is an algebraic completely integrable system. We study the problem of constructing elementary (one- and two-point) BTs for this system. In section 3 we construct an elementary (one-point) BT which gives an exact discretization of a specific continuous flow. By making a two-point composite map in section 7 we are then able to discretize any of the independent commuting flows with the Hamiltonians from the spectral curve of the 2×2 Lax matrix.

2. Gaudin magnet

The $sl(2)$ Gaudin magnet is derived from the Lax matrix

$$L(u) = \sum_{j=1}^{n} \frac{1}{u-a_j} \begin{pmatrix} s_j^3 & s_j^- \\ s_j^+ & -s_j^3 \end{pmatrix} + \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} A(u) & B(u) \\ C(u) & -A(u) \end{pmatrix} \tag{2.1}$$

$$A(u) = \alpha + \sum_{j=1}^{n} \frac{s_j^3}{u-a_j} \qquad B(u) = \sum_{j=1}^{n} \frac{s_j^-}{u-a_j} \qquad C(u) = \sum_{j=1}^{n} \frac{s_j^+}{u-a_j}. \tag{2.2}$$

Local variables in this model are generators of the direct sum of n $sl(2)$ spins, s_j^3, $s_j^\pm$, $j = 1, \ldots, n$, with the following Poisson brackets:

$$\{s_j^3, s_k^\pm\} = \mp \mathrm{i}\delta_{jk} s_k^\pm \qquad \{s_j^+, s_k^-\} = -2\mathrm{i}\delta_{jk} s_k^3. \tag{2.3}$$

We denote Casimir operators (spins) as s_j:

$$s_j^2 = (s_j^3)^2 + s_j^+ s_j^-. \tag{2.4}$$

Fixing Casimirs s_j we go to a symplectic leaf where the Poisson bracket is non-degenerate, so that the symplectic manifold is a collection of n spheres.

Let us also introduce the total spin $\vec{J}$ which will be used later, as follows:

$$J_3 = \sum_{j=1}^{n} s_j^3 \qquad J_+ = \sum_{j=1}^{n} s_j^+ \qquad J_- = \sum_{j=1}^{n} s_j^-. \tag{2.5}$$

The Lax matrix (2.1) satisfies the linear r-matrix Poisson algebra,

$$\{L_1(u), L_2(v)\} = [r(u-v), L_1(u) + L_2(v)] \tag{2.6}$$

with the permutation matrix as the r-matrix

$$r(u-v) = \frac{\mathrm{i}}{u-v} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.7}$$

Here we use standard notations L_1 and L_2 for tensor products:

$$L_1(u) = L(u) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad L_2(v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes L(v). \tag{2.8}$$

Equation (2.6) is equivalent to the following Poisson brackets for the rational functions $A(u)$, $B(u)$ and $C(u)$:

$$\{A(u), A(v)\} = \{B(u), B(v)\} = \{C(u), C(v)\} = 0 \tag{2.9}$$

$$\{A(u), B(v)\} = \frac{\mathrm{i}}{u-v}(B(v) - B(u)) \tag{2.10}$$

$$\{A(u), C(v)\} = \frac{-\mathrm{i}}{u-v}(C(v) - C(u)) \tag{2.11}$$

$$\{C(u), B(v)\} = \frac{-2\mathrm{i}}{u-v}(A(v) - A(u)). \tag{2.12}$$

The spectral curve Γ,

$$\Gamma : \det(L(u) - v) = 0 \tag{2.13}$$

is a hyperelliptic, genus $n-1$ curve,

$$v^2 = A^2(u) + B(u)C(u) = \alpha^2 + \sum_{j=1}^{n}\left(\frac{H_j}{u-a_j} + \frac{s_j^2}{(u-a_j)^2}\right) \tag{2.14}$$

with the Hamiltonians (integrals of motion) H_j of the form

$$H_j = \sum_{k\neq j}\frac{2s_j^3s_k^3 + s_j^+s_k^- + s_j^-s_k^+}{a_j - a_k} + 2\alpha s_j^3. \tag{2.15}$$

These are integrals of motion, or Hamiltonians, of the $sl(2)$ Gaudin magnet, which are Poisson commuting:

$$\{H_j, H_k\} = 0 \qquad j, k = 1, \ldots, n. \tag{2.16}$$

Notice that there is one linear integral:

$$\sum_{j=1}^{n} H_j = 2\alpha J_3. \tag{2.17}$$

We can bring the curve Γ into the canonical form by scaling the variable $v \mapsto \hat{v}$:

$$\hat{v} = v\prod_{j=1}^{n}(u - a_j). \tag{2.18}$$

The equation of the curve becomes

$$\begin{aligned}\hat{v}^2 &= \left[\alpha^2 + \sum_{j=1}^{n}\left(\frac{H_j}{u-a_j} + \frac{s_j^2}{(u-a_j)^2}\right)\right]\prod_{j=1}^{n}(u-a_j)^2 \\ &= \alpha^2u^{2n} + f_1u^{2n-1} + f_2u^{2n-2} + \cdots + f_{2n}.\end{aligned} \tag{2.19}$$

When $\alpha = 0$ the genus of the curve drops to $n-2$, because $f_1 = 0$ in such a case. The Gaudin magnet then becomes $sl(2)$ invariant: as well as integrals (2.15) all three components of the total spin $\vec{J}$ are integrals too,

$$\alpha = 0 : \{H_j, J_k\} = 0 \qquad j = 1, \ldots, n \qquad k = 1, 2, 3. \tag{2.20}$$

We do not consider this case, but concentrate on the generic case of $\alpha \neq 0$ when there is only one linear integral $f_1 = 2\alpha(J_3 - \alpha\sum_{j=1}^{n} a_j)$. The latter case is called the generic (diagonal) case of the $sl(2)$ *XXX* Gaudin magnet. It is known that all its flows are linearized on the generalized Jacobian of the hyperelliptic curve (2.19): see references in [9].

3. One-point basic map

The $sl(2)$ Gaudin magnet with the 2×2 Lax matrix (2.1) is within the class of systems that was considered recently in [9], namely it belongs to the (even) case of the generalized Jacobian. Hence, its BTs can be extracted from that paper. However, we want to present here an independent derivation of those BTs as well as give a more detailed exposition of their various properties. The reader is referred to [9] for explanation of the geometric meaning of BTs.

A BT should act on the Lax matrix as a similarity transform:

$$L(u) \mapsto M(u)L(u)M(u)^{-1} \qquad \forall u \tag{3.1}$$

with some non-degenerate 2×2 matrix $M(u)$, simply because a BT should preserve the spectrum of $L(u)$.

We introduce tilde ($\tilde{\ }$) notation for the up-dated variables:

$$\tilde{L}(u) = \sum_{j=1}^{n} \frac{1}{u - a_j} \begin{pmatrix} \tilde{s}_j^3 & \tilde{s}_j^- \\ \tilde{s}_j^+ & -\tilde{s}_j^3 \end{pmatrix} + \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \tilde{A}(u) & \tilde{B}(u) \\ \tilde{C}(u) & -\tilde{A}(u) \end{pmatrix} \tag{3.2}$$

$$\tilde{A}(u) = \alpha + \sum_{j=1}^{n} \frac{\tilde{s}_j^3}{u - a_j} \qquad \tilde{B}(u) = \sum_{j=1}^{n} \frac{\tilde{s}_j^-}{u - a_j} \qquad \tilde{C}(u) = \sum_{j=1}^{n} \frac{\tilde{s}_j^+}{u - a_j} \tag{3.3}$$

$$\{\tilde{s}_j^3, \tilde{s}_k^\pm\} = \mp \mathrm{i}\delta_{jk}\tilde{s}_k^\pm \qquad \{\tilde{s}_j^+, \tilde{s}_k^-\} = -2\mathrm{i}\delta_{jk}\tilde{s}_k^3. \tag{3.4}$$

We are looking for a Poisson map that intertwines two Lax matrices $L(u)$ and $\tilde{L}(u)$:

$$M(u)L(u) = \tilde{L}(u)M(u) \qquad \forall u. \tag{3.5}$$

Because spins s_j, $j = 1, \ldots, n$, appear as coefficients of the curve, they are not changed by the map, i.e. $\tilde{s}_j = s_j$. Hence, we can talk about a symplectomorphism ($s_j = \text{const}$) instead of a Poisson map.

Now we should choose an ansatz for the dependence of the matrix $M(u)$ on the spectral parameter u. Let us fix the simplest case of a *linear* function:

$$M(u) = M_1 u + M_0. \tag{3.6}$$

Taking the limit $u \to \infty$ in (3.5) we conclude that M_1 must be diagonal. Moreover, the most elementary (one-point) BT should correspond to the case when $\det M(u)$ has only one zero $u = \lambda$, which will lead to having only one Bäcklund parameter (cf the spectrality property in [8, 10]). So, we should choose either

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{or} \qquad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.7}$$

We will consider the first case, as the second one will produce a similar BT (moving in discrete time in the positive and in the negative direction). Finally, we arrive at the following parametrization of the unknown matrix $M(u)$:

$$M(u) = \begin{pmatrix} u - \lambda + pq/\gamma & p \\ q & \gamma \end{pmatrix} \tag{3.8}$$

with the c-number determinant

$$\det M(u) = \gamma(u - \lambda). \tag{3.9}$$

Here the variables p and q are indeterminate dynamical variables, but λ and γ are c-number Bäcklund parameters. ($M(u)$ comes from a L-operator of the quadratic r-matrix algebra whose det is a Casimir, see [12].)

Comparing asymptotics in u in both sides of (3.5) we readily get

$$p = \frac{J_-}{2\alpha} \qquad q = \frac{\tilde{J}_+}{2\alpha}. \tag{3.10}$$

If we want an explicit single-valued map from $L(u)$ to $\tilde{L}(u)$

$$L(u) \mapsto \tilde{L}(u) = M(u)L(u)M^{-1}(u) \tag{3.11}$$

then we must express $M(u)$, and therefore p and q, in terms of the old variables, i.e. the entries of $L(u)$ only. There is, however, a problem, since from (3.10) we have only the expression for the p, but the variable q is given in terms of the new, and therefore *unknown*, variable $\tilde{J}_+$. To overcome this difficulty we use an extra piece of data, namely the spectrality. As well as equation (3.5) that our map satisfies, it will be parametrized by the point P on the curve Γ,

$$P = (\lambda, \mu) \in \Gamma. \tag{3.12}$$

Notice that there are two points on the curve Γ, $P = (\lambda, +\mu)$ and $Q = (\lambda, -\mu)$, corresponding to a fixed λ and sitting one above the other because of the hyperelliptic involution:

$$(\lambda, \mu) \in \Gamma : \ \det(L(\lambda) - \mu) = 0 \Leftrightarrow \mu^2 + \det(L(\lambda)) = 0. \tag{3.13}$$

Because $\det(M(\lambda)) = 0$ the matrix $M(\lambda)$ has a one-dimensional kernel

$$M(\lambda)\Omega = \begin{pmatrix} pq/\gamma & p \\ q & \gamma \end{pmatrix}\Omega = 0 \qquad \Omega = \begin{pmatrix} \gamma \\ -q \end{pmatrix}. \tag{3.14}$$

The equality

$$M(\lambda)L(\lambda)\Omega = \tilde{L}(\lambda)M(\lambda)\Omega = 0 \tag{3.15}$$

implies that $L(\lambda)\Omega \sim \Omega$, so that Ω is an eigenvector of $L(\lambda)$. Let us fix the corresponding point of the spectrum as $P = (\lambda, \mu)$:

$$\begin{pmatrix} A(\lambda) - \mu & B(\lambda) \\ C(\lambda) & -A(\lambda) - \mu \end{pmatrix}\begin{pmatrix} \gamma \\ -q \end{pmatrix} = 0. \tag{3.16}$$

This gives us the formula for the variable q:

$$q = \gamma\,\frac{A(\lambda) - \mu}{B(\lambda)} = -\gamma\,\frac{C(\lambda)}{A(\lambda) + \mu}. \tag{3.17}$$

The two last expressions are equivalent since $(\lambda, \mu) \in \Gamma$.

Now, the formulae (3.8), (3.10), (3.11), and (3.17) give a one-point Poisson integrable map ($\equiv$ one-point BT) from $L(u)$ to $\tilde{L}(u)$. The map is parametrized by one point (λ, μ) on the spectral curve Γ (and by an extra parameter γ).

Explicitly, it reads

$$\tilde{A}(u) = \frac{\gamma(u - \lambda + 2pq/\gamma)A(u) - q(u - \lambda + pq/\gamma)B(u) + p\gamma C(u)}{\gamma(u - \lambda)} \tag{3.18}$$

$$\tilde{B}(u) = \frac{(u - \lambda + pq/\gamma)^2 B(u) - 2p(u - \lambda + pq/\gamma)A(u) - p^2 C(u)}{\gamma(u - \lambda)} \tag{3.19}$$

$$\tilde{C}(u) = \frac{\gamma^2 C(u) + 2q\gamma A(u) - q^2 B(u)}{\gamma(u - \lambda)}. \tag{3.20}$$

Equating residues at $u = a_j$ in both sides of the above equations, we obtain the map in terms of the local spin variables:

$$\tilde{s}_j^3 = \frac{\gamma(a_j - \lambda + 2pq/\gamma)s_j^3 - q(a_j - \lambda + pq/\gamma)s_j^- + p\gamma s_j^+}{\gamma(a_j - \lambda)} \tag{3.21}$$

$$\tilde{s}_j^- = \frac{(a_j - \lambda + pq/\gamma)^2 s_j^- - 2p(a_j - \lambda + pq/\gamma)s_j^3 - p^2 s_j^+}{\gamma(a_j - \lambda)} \tag{3.22}$$

$$\tilde{s}_j^+ = \frac{\gamma^2 s_j^+ + 2q\gamma s_j^3 - q^2 s_j^-}{\gamma(a_j - \lambda)}. \tag{3.23}$$

Recall that α, a_j and s_j, $j = 1, \ldots, n$, are parameters of the model; γ and λ are parameters of the map and variables p and q are as follows:

$$p = \frac{J_-}{2\alpha} \qquad q = \gamma\,\frac{A(\lambda) - \mu}{B(\lambda)} = -\gamma\,\frac{C(\lambda)}{A(\lambda) + \mu} \tag{3.24}$$

$$\mu^2 = \alpha^2 + \sum_{j=1}^{n}\left(\frac{H_j}{\lambda - a_j} + \frac{s_j^2}{(\lambda - a_j)^2}\right) \tag{3.25}$$

$$H_j = \sum_{k\neq j}\frac{2s_j^3 s_k^3 + s_j^+ s_k^- + s_j^- s_k^+}{a_j - a_k} + 2\alpha s_j^3. \tag{3.26}$$

4. BT as a discrete-time map

In this section we show that the BT constructed above can be seen as a time discretization of a specific Hamiltonian flow where the parameter λ plays the role of inversion of the time step.

Consider the limit $\lambda \to \infty$. Then

$$\mu = \alpha + \mathrm{O}\left(\frac{1}{\lambda}\right). \tag{4.1}$$

Assume that

$$\gamma = -\lambda + \gamma_0 + \mathrm{O}\left(\frac{1}{\lambda}\right). \tag{4.2}$$

Then we have the following expansion for the matrix $M(u)$:

$$M(u) = -\lambda\left(1 - \frac{1}{2\lambda}M_0(u)\right) + \mathrm{O}\left(\frac{1}{\lambda}\right). \tag{4.3}$$

The equation of the map, $M(u)L(u) = \tilde{L}(u)M(u)$, turns in this limit into the Lax pair of a continuous flow:

$$\dot{L}(u) = [L(u), M_0(u)] \qquad M_0(u) = \begin{pmatrix} u - \gamma_0 & J_-/\alpha \\ J_+/\alpha & -u + \gamma_0 \end{pmatrix} \tag{4.4}$$

where $1/(2\lambda)$ is a time step and $\dot{L}(u) \equiv \lim_{\lambda\to\infty} 2\lambda(\tilde{L}(u) - L(u))$ is the time derivative.

The flow (4.4) is a Hamiltonian flow

$$\dot{L}(u) = \{H, L(u)\} \tag{4.5}$$

with the Hamiltonian function H as

$$H = \frac{\mathrm{i}}{\alpha}\left(J_+ J_- + 2\alpha\sum_{j=1}^{n}(a_j - \gamma_0)s_j^3\right). \tag{4.6}$$

Therefore, the constructed BT is a two-parameter (λ, γ) time discretization of this continuous flow.

5. Symplecticity

In this section we give a simple proof of symplecticity of the constructed map by finding an explicit generating function of the corresponding canonical transformation from the old to new variables.

First, because the spin variables (Casimirs) do not change,

$$s_j^2 = (s_j^3)^2 + s_j^+ s_j^- = (\tilde{s}_j^3)^2 + \tilde{s}_j^+ \tilde{s}_j^- \tag{5.1}$$

we can exclude the variables s_j^+ and $\tilde{s}_j^-$, $j = 1, \ldots, n$,

$$s_j^+ = \frac{s_j^2 - (s_j^3)^2}{s_j^-} \qquad \tilde{s}_j^- = \frac{s_j^2 - (\tilde{s}_j^3)^2}{\tilde{s}_j^+} \tag{5.2}$$

expressing everything in terms of $2n$ 'canonical' variables $(s_j^3, s_j^-)_{j=1}^n$ and $(\tilde{s}_j^3, \tilde{s}_j^+)_{j=1}^n$ with the following Poisson brackets:

$$\{s_j^3, s_k^-\} = \mathrm{i}\delta_{jk} s_k^- \qquad \{\tilde{s}_j^3, \tilde{s}_k^+\} = -\mathrm{i}\delta_{jk} \tilde{s}_k^+. \tag{5.3}$$

We want to represent our BT as a canonical transformation defined by the generating function $F(\tilde{s}^+|s^-) \equiv F_{\lambda,\gamma}(\tilde{s}^+|s^-)$ such that

$$s_j^3 = \mathrm{i} s_j^- \frac{\partial F(\tilde{s}^+|s^-)}{\partial s_j^-} \qquad \tilde{s}_j^3 = \mathrm{i}\tilde{s}_j^+ \frac{\partial F(\tilde{s}^+|s^-)}{\partial \tilde{s}_j^+}. \tag{5.4}$$

Notice that we have chosen the arguments of the generating function as $(\tilde{s}_j^+|s_j^-)_{j=1}^n$. Because the symplecticity property does not depend on the choice of the arguments of its generating function, these arguments are fixed in order to get a simpler expression for the function $F_{\lambda,\gamma}$ (recall that the variables p and q of (3.10) depend exactly on these variables).

Rewrite now the equations of the map (3.21)–(3.23) in the form

$$\left(\gamma s_j^3 - \frac{\tilde{J}_+}{2\alpha} s_j^-\right)^2 + \gamma(a_j - \lambda)\tilde{s}_j^+ s_j^- - \gamma^2 s_j^2 = 0 \tag{5.5}$$

$$\left(\gamma \tilde{s}_j^3 - \frac{J_-}{2\alpha} \tilde{s}_j^+\right)^2 + \gamma(a_j - \lambda)\tilde{s}_j^+ s_j^- - \gamma^2 s_j^2 = 0. \tag{5.6}$$

Resolving them with respect to s_j^3 and $\tilde{s}_j^3$, we obtain

$$s_j^3 = \frac{\tilde{J}_+}{2\alpha\gamma} s_j^- + z_j \qquad \tilde{s}_j^3 = \frac{J_-}{2\alpha\gamma} \tilde{s}_j^+ + z_j \tag{5.7}$$

$$z_j^2 = s_j^2 - \frac{a_j - \lambda}{\gamma} \tilde{s}_j^+ s_j^- \qquad j = 1, \ldots, n. \tag{5.8}$$

It is now easy to check that the function

$$F_{\lambda,\gamma}(\tilde{s}^+|s^-) = -\mathrm{i}\frac{\tilde{J}_+ J_-}{2\alpha\gamma} - \mathrm{i}\sum_{j=1}^n \left(2z_j + s_j \log \frac{z_j - s_j}{z_j + s_j}\right) \tag{5.9}$$

satisfies equations (5.4). Thereby symplecticity of the map is proven.

6. Spectrality

The map depends on two parameters, λ and γ. Let us first concentrate on its λ-dependence.

Spectrality, which was introduced in [8], is an interesting property of BTs. It usually holds for any BT which has a parameter. Technically, this means that the components of the point $P = (\lambda, \mu) \in \Gamma$ which parametrizes the map are conjugated variables, namely

$$\mu = \frac{\partial F_{\lambda,\gamma}(\tilde{s}^+|s^-)}{\partial \lambda}. \tag{6.1}$$

To prove this formula, use (3.10) and (3.17) to find the formula for the μ,

$$\mu = A(\lambda) - \frac{\tilde{J}_+}{2\alpha\gamma} B(\lambda). \tag{6.2}$$

Now, with the help of (5.7) and (5.9) we can easily check the needed formula for the spectrality property (6.1).

A new (compared with [8]) observation is that there is also an analogous property with respect to the parameter γ, only now it is the integral J_3 that plays the role of the conjugated variable:

$$J_3 = -\gamma \frac{\partial F_{\lambda,\gamma}(\tilde{s}^+|s^-)}{\partial \gamma}. \tag{6.3}$$

The proof is very simple, once we notice that (5.7) entails

$$J_3 = \frac{\tilde{J}_+ J_-}{2\alpha\gamma} + \sum_{j=1}^{n} z_j. \tag{6.4}$$

To conclude this section we remark that, because of the second 'spectrality' property (6.3), which was somehow built into the BT from the very beginning, one can recover the generating function of the map just by taking one integral,

$$F_{\lambda,\gamma}(\tilde{s}^+|s^-) = \int^{\gamma} \left(-\frac{J_3}{\gamma}\right) \mathrm{d}\gamma + \text{const} \tag{6.5}$$

without needing to solve the system of $2n$ differential equations (5.7).

7. Inverse map and a two-point map

In this section we first construct the inverse BT and then use it to derive the two-point BT which will be a composition of the direct map parametrized by the point $P_1 = (\lambda_1, \mu_1) \in \Gamma$ and the inverse map parametrized by the point $Q_2 = (\lambda_2, -\mu_2) \in \Gamma$.

7.1. The inversion of the Bäcklund transformation

Let us call the direct map B_P. The inverse map acts from $\tilde{L}(u)$ to $L(u)$. We can rewrite the equations for B_P in the inverse form

$$M^\wedge(u)\tilde{L}(u) = L(u)M^\wedge(u) \qquad M^\wedge(u) = \begin{pmatrix} \gamma & -p \\ -q & u - \lambda + pq/\gamma \end{pmatrix}. \tag{7.1}$$

To define the inverse map we must find expressions for the co-factor matrix $M^\wedge(u)$, or for the variables p and q, in terms of $\tilde{}$-variables, i.e. in terms of the entries of $\tilde{L}(u)$. We already have the expressions of (3.10),

$$p = \frac{J_-}{2\alpha} \qquad q = \frac{\tilde{J}_+}{2\alpha} \tag{7.2}$$

which define q. To obtain the formula for the variable p we again use the spectrality property. The matrix $M^{\wedge}(\lambda)$ has a one-dimensional kernel $\tilde{\Omega}$,

$$M^{\wedge}(\lambda)\tilde{\Omega} = \begin{pmatrix} \gamma & -p \\ -q & pq/\gamma \end{pmatrix} \tilde{\Omega} = 0 \qquad \tilde{\Omega} = \begin{pmatrix} p \\ \gamma \end{pmatrix}. \tag{7.3}$$

The main difference compared with the formulae of the direct map is that the inverse map will be parametrized by the point $Q = (\lambda, -\mu) \in \Gamma$, not $P = (\lambda, \mu) \in \Gamma$. Therefore, $\tilde{\Omega}$ is an eigenvector of the matrix $\tilde{L}(u)$ with the eigenvalue $Q = (\lambda, -\mu)$:

$$\begin{pmatrix} \tilde{A}(\lambda) + \mu & \tilde{B}(\lambda) \\ \tilde{C}(\lambda) & -\tilde{A}(\lambda) + \mu \end{pmatrix} \begin{pmatrix} p \\ \gamma \end{pmatrix} = 0. \tag{7.4}$$

This gives us the formula needed for the variable p,

$$p = \gamma \, \frac{\tilde{A}(\lambda) - \mu}{\tilde{C}(\lambda)} = -\gamma \, \frac{\tilde{B}(\lambda)}{\tilde{A}(\lambda) + \mu}. \tag{7.5}$$

To prove that this does indeed give the inverse map, we have to show that the two formulae, (3.17) and (7.5), in fact define the same variable μ:

$$\mu = \tilde{A}(\lambda) - \frac{p}{\gamma} \tilde{C}(\lambda) \stackrel{?}{=} A(\lambda) - \frac{q}{\gamma} B(\lambda). \tag{7.6}$$

It is easy to see that this equation is the (11)-element of the matrix identity:

$$M^{\wedge}(\lambda)\tilde{L}(\lambda) = L(\lambda)M^{\wedge}(\lambda). \tag{7.7}$$

We denote as $\mathcal{B}_Q$ the map which is inverse to the map B_P. Generally speaking, we have constructed four different maps, B_P, B_Q, $\mathcal{B}_Q$, and $\mathcal{B}_P$, with two pairs of maps which are inverse to each other:

$$\mathcal{B}_Q \circ B_P = B_P \circ \mathcal{B}_Q = \mathcal{B}_P \circ B_Q = B_Q \circ \mathcal{B}_P = \mathrm{Id}. \tag{7.8}$$

7.2. The two-point map B_{P_1,Q_2}

We now construct a composite map which is a product of the map $B_{P_1} \equiv B_{(\lambda_1,\mu_1)}$ and $\mathcal{B}_{Q_2} \equiv \mathcal{B}_{(\lambda_2,-\mu_2)}$:

$$B_{P_1,Q_2} = \mathcal{B}_{Q_2} \circ B_{P_1}. \tag{7.9}$$

The second parameter of the basic map, namely the γ, is taken to be the same in both maps, so $\gamma_1 = \gamma_2$. Obviously, when $\lambda_1 = \lambda_2$ (and $\mu_1 = \mu_2$) this composite map will turn into an identity map.

The first map B_{P_1} reads as follows:

$$M_1(u)L(u) = \tilde{L}(u)M_1(u) \qquad M_1(u) = \begin{pmatrix} u - \lambda_1 + p_1 q_1/\gamma & p_1 \\ q_1 & \gamma \end{pmatrix} \tag{7.10}$$

where the formulae for the variables p_1 and q_1 are

$$p_1 = \frac{J_-}{2\alpha} = \gamma \, \frac{\tilde{A}(\lambda_1) - \mu_1}{\tilde{C}(\lambda_1)} = -\gamma \, \frac{\tilde{B}(\lambda_1)}{\tilde{A}(\lambda_1) + \mu_1} \tag{7.11}$$

$$q_1 = \frac{\tilde{J}_+}{2\alpha} = \gamma \, \frac{A(\lambda_1) - \mu_1}{B(\lambda_1)} = -\gamma \, \frac{C(\lambda_1)}{A(\lambda_1) + \mu_1}. \tag{7.12}$$

The second map B_{Q_2} reads as follows:

$$M_2(u)\tilde{L}(u) = \tilde{\tilde{L}}(u)M_2(u) \qquad M_2(u) = \begin{pmatrix} \gamma & -p_2 \\ -q_2 & u - \lambda_2 + p_2 q_2/\gamma \end{pmatrix} \tag{7.13}$$

where the formulae for the variables p_2 and q_2 are

$$p_2 = \frac{\tilde{\tilde{J}}_-}{2\alpha} = \gamma\,\frac{\tilde{A}(\lambda_2) - \mu_2}{\tilde{C}(\lambda_2)} = -\gamma\,\frac{\tilde{B}(\lambda_2)}{\tilde{A}(\lambda_2) + \mu_2} \tag{7.14}$$

$$q_2 = \frac{\tilde{J}_+}{2\alpha} = \gamma\,\frac{\tilde{\tilde{A}}(\lambda_2) - \mu_2}{\tilde{\tilde{B}}(\lambda_2)} = -\gamma\,\frac{\tilde{\tilde{C}}(\lambda_2)}{\tilde{\tilde{A}}(\lambda_2) + \mu_2}. \tag{7.15}$$

Notice that q_1 is equal to q_2, hence we omit the sub-index, $q_1 = q_2 = q$.

The composite map B_{P_1,Q_2} acts from $L(u)$ to $\tilde{\tilde{L}}(u)$,

$$M(u)L(u) = \tilde{\tilde{L}}(u)M(u) \tag{7.16}$$

$$M(u) = \frac{1}{\gamma}\,M_2(u)M_1(u) = \begin{pmatrix} u - \lambda_1 + \frac{q}{\gamma}\,(p_1 - p_2) & p_1 - p_2 \\ \frac{q}{\gamma}\,(\lambda_1 - \lambda_2 - \frac{q}{\gamma}\,(p_1 - p_2)) & u - \lambda_2 - \frac{q}{\gamma}\,(p_1 - p_2) \end{pmatrix}. \tag{7.17}$$

In order to get rid of the intermediate $\tilde{}$-variables, we use the spectrality property with respect to two points, $P_1 = (\lambda_1, \mu_1)$ and $Q_2 = (\lambda_2, -\mu_2)$. Obviously, both spectralities are still valid after composing the maps. For the point P_1 we get the following equations:

$$M(\lambda_1)\Omega_1 = 0 \qquad \Omega_1 = \begin{pmatrix} \gamma \\ -q \end{pmatrix} \qquad L(\lambda_1)\Omega_1 = \mu_1\Omega_1$$

$$\Rightarrow q = \gamma\,\frac{A(\lambda_1) - \mu_1}{B(\lambda_1)} = -\gamma\,\frac{C(\lambda_1)}{A(\lambda_1) + \mu_1} \tag{7.18}$$

$$M^\wedge(\lambda_1)\tilde{\tilde{\Omega}}_1 = 0 \qquad \tilde{\tilde{\Omega}}_1 = \begin{pmatrix} p_1 - p_2 \\ \lambda_1 - \lambda_2 - \frac{q}{\gamma}\,(p_1 - p_2) \end{pmatrix} \qquad \tilde{\tilde{L}}(\lambda_1)\tilde{\tilde{\Omega}}_1 = -\mu_1\tilde{\tilde{\Omega}}_1$$

$$\Rightarrow p_1 - p_2 = \frac{\gamma(\lambda_2 - \lambda_1)\tilde{\tilde{B}}(\lambda_1)}{\gamma(\tilde{\tilde{A}}(\lambda_1) + \mu_1) - q\tilde{\tilde{B}}(\lambda_1)} = \frac{\gamma(\lambda_1 - \lambda_2)(\tilde{\tilde{A}}(\lambda_1) - \mu_1)}{q(\tilde{\tilde{A}}(\lambda_1) - \mu_1) + \gamma\tilde{\tilde{C}}(\lambda_1)}. \tag{7.19}$$

For the point Q_2 we get the second set of equations:

$$M(\lambda_2)\Omega_2 = 0 \qquad \Omega_2 = \begin{pmatrix} p_1 - p_2 \\ \lambda_1 - \lambda_2 - \frac{q}{\gamma}\,(p_1 - p_2) \end{pmatrix} \qquad L(\lambda_2)\Omega_2 = -\mu_2\Omega_2$$

$$\Rightarrow p_1 - p_2 = \frac{\gamma(\lambda_2 - \lambda_1)B(\lambda_2)}{\gamma(A(\lambda_2) + \mu_2) - qB(\lambda_2)} = \frac{\gamma(\lambda_1 - \lambda_2)(A(\lambda_2) - \mu_2)}{q(A(\lambda_2) - \mu_2) + \gamma C(\lambda_2)} \tag{7.20}$$

$$M^\wedge(\lambda_2)\tilde{\tilde{\Omega}}_2 = 0 \qquad \tilde{\tilde{\Omega}}_2 = \begin{pmatrix} \gamma \\ -q \end{pmatrix} \qquad \tilde{\tilde{L}}(\lambda_2)\tilde{\tilde{\Omega}}_2 = \mu_2\tilde{\tilde{\Omega}}_2$$

$$\Rightarrow q = \gamma\,\frac{\tilde{\tilde{A}}(\lambda_2) - \mu_2}{\tilde{\tilde{B}}(\lambda_2)} = -\gamma\,\frac{\tilde{\tilde{C}}(\lambda_2)}{\tilde{\tilde{A}}(\lambda_2) + \mu_2}. \tag{7.21}$$

Equations (7.18) and (7.21) are already known to us (cf (7.12) and (7.15)). The formulae (7.19) and (7.20) for the variable $p_1 - p_2$ are new. They are equivalent to the formulae (7.11) and (7.14) expressed in terms of entries of $L(u)$ and $\tilde{\tilde{L}}(u)$.

Concluding, we have constructed a two-point BT which is factorized to two one-point BTs and which is explicitly given, together with its inverse, by the formulae:

$$M(u)L(u) = \tilde{\tilde{L}}(u)M(u) \qquad M(u) = \begin{pmatrix} u - \lambda_1 + xX & X \\ -x^2X + (\lambda_1 - \lambda_2)x & u - \lambda_2 - xX \end{pmatrix} \tag{7.22}$$

where

$$x := \frac{A(\lambda_1) - \mu_1}{B(\lambda_1)} = -\frac{C(\lambda_1)}{A(\lambda_1) + \mu_1} = \frac{\tilde{\tilde{A}}(\lambda_2) - \mu_2}{\tilde{\tilde{B}}(\lambda_2)} = -\frac{\tilde{\tilde{C}}(\lambda_2)}{\tilde{\tilde{A}}(\lambda_2) + \mu_2} \tag{7.23}$$

$$X := \frac{(\lambda_2 - \lambda_1)B(\lambda_1)B(\lambda_2)}{B(\lambda_1)(A(\lambda_2) + \mu_2) - B(\lambda_2)(A(\lambda_1) - \mu_1)} \tag{7.24}$$

$$= \frac{(\lambda_1 - \lambda_2)B(\lambda_1)(A(\lambda_2) - \mu_2)}{(A(\lambda_1) - \mu_1)(A(\lambda_2) - \mu_2) + B(\lambda_1)C(\lambda_2)} \tag{7.25}$$

$$= \frac{(\lambda_2 - \lambda_1)B(\lambda_2)(A(\lambda_1) + \mu_1)}{(A(\lambda_1) + \mu_1)(A(\lambda_2) + \mu_2) + B(\lambda_2)C(\lambda_1)} \tag{7.26}$$

$$= \frac{(\lambda_1 - \lambda_2)(A(\lambda_1) + \mu_1)(A(\lambda_2) - \mu_2)}{(A(\lambda_1) + \mu_1)C(\lambda_2) - (A(\lambda_2) - \mu_2)C(\lambda_1)} \tag{7.27}$$

$$= \frac{(\lambda_2 - \lambda_1)\tilde{\tilde{B}}(\lambda_2)\tilde{\tilde{B}}(\lambda_1)}{\tilde{\tilde{B}}(\lambda_2)(\tilde{\tilde{A}}(\lambda_1) + \mu_1) - \tilde{\tilde{B}}(\lambda_1)(\tilde{\tilde{A}}(\lambda_2) - \mu_2)} \tag{7.28}$$

$$= \frac{(\lambda_1 - \lambda_2)\tilde{\tilde{B}}(\lambda_2)(\tilde{\tilde{A}}(\lambda_1) - \mu_1)}{(\tilde{\tilde{A}}(\lambda_2) - \mu_2)(\tilde{\tilde{A}}(\lambda_1) - \mu_1) + \tilde{\tilde{B}}(\lambda_2)\tilde{\tilde{C}}(\lambda_1)} \tag{7.29}$$

$$= \frac{(\lambda_2 - \lambda_1)\tilde{\tilde{B}}(\lambda_1)(\tilde{\tilde{A}}(\lambda_2) + \mu_2)}{(\tilde{\tilde{A}}(\lambda_2) + \mu_2)(\tilde{\tilde{A}}(\lambda_1) + \mu_1) + \tilde{\tilde{B}}(\lambda_1)\tilde{\tilde{C}}(\lambda_2)} \tag{7.30}$$

$$= \frac{(\lambda_1 - \lambda_2)(\tilde{\tilde{A}}(\lambda_2) + \mu_2)(\tilde{\tilde{A}}(\lambda_1) - \mu_1)}{(\tilde{\tilde{A}}(\lambda_2) + \mu_2)\tilde{\tilde{C}}(\lambda_1) - (\tilde{\tilde{A}}(\lambda_1) - \mu_1)\tilde{\tilde{C}}(\lambda_2)}. \tag{7.31}$$

The above formulae give several equivalent expressions for the variables x and X since the points (λ_1, μ_1) and $(\lambda_2, -\mu_2)$ belong to the spectral curve Γ, i.e. are bound by the following relations:

$$\mu_1^2 = A^2(\lambda_1) + B(\lambda_1)C(\lambda_1) \qquad \mu_2^2 = A^2(\lambda_2) + B(\lambda_2)C(\lambda_2) \tag{7.32}$$

$$\mu_1^2 = \tilde{\tilde{A}}^2(\lambda_1) + \tilde{\tilde{B}}(\lambda_1)\tilde{\tilde{C}}(\lambda_1) \qquad \mu_2^2 = \tilde{\tilde{A}}^2(\lambda_2) + \tilde{\tilde{B}}(\lambda_2)\tilde{\tilde{C}}(\lambda_2). \tag{7.33}$$

7.3. Two-point map as a discrete-time map

We will see in this section that the two-point map constructed above is a one-parameter, λ_1, time discretization of a family of flows parametrized by the point $Q_2 = (\lambda_2, -\mu_2)$, with the difference $\lambda_1 - \lambda_2$ playing the role of the time step.

Indeed, consider the limit $\lambda_1 \to \lambda_2$,

$$\lambda_1 = \lambda_2 + \varepsilon \qquad \varepsilon \to 0. \tag{7.34}$$

It is easy to see from the formulae of the previous section that

$$x = x_0 + \mathrm{O}(\varepsilon) \qquad x_0 = \frac{A(\lambda_2) - \mu_2}{B(\lambda_2)} = -\frac{C(\lambda_2)}{A(\lambda_2) + \mu_2} \tag{7.35}$$

and

$$X = \varepsilon X_0 + \mathrm{O}(\varepsilon^2) \qquad X_0 = -\frac{B(\lambda_2)}{2\mu_2}. \tag{7.36}$$

Then we derive that the matrix $M(u)$ has the following asymptotics:

$$M(u) = (u - \lambda_2)\left(1 - \frac{\varepsilon}{2\mu_2(u - \lambda_2)}\begin{pmatrix} A(\lambda_2) + \mu_2 & B(\lambda_2) \\ C(\lambda_2) & -A(\lambda_2) + \mu_2 \end{pmatrix}\right) + \mathrm{O}(\varepsilon^2). \tag{7.37}$$

If we now define the time derivative $\dot{L}(u)$ as

$$\dot{L}(u) = \lim_{\varepsilon \to 0} \frac{\tilde{\tilde{L}}(u) - L(u)}{\varepsilon} \tag{7.38}$$

then in the limit we obtain from the equation of the map, $M(u)L(u) = \tilde{\tilde{L}}(u)M(u)$, the Lax equation for a corresponding continuous flow that our BT discretizes, namely

$$\dot{L}(u) = \left[L(u), \frac{L(\lambda_2)}{2\mu_2(u - \lambda_2)} \right]. \tag{7.39}$$

This is a Hamiltonian flow with μ_2,

$$\mu_2 = \sqrt{A^2(\lambda_2) + B(\lambda_2)C(\lambda_2)} = \sqrt{\alpha^2 + \sum_{j=1}^{n} \left(\frac{H_j}{\lambda_2 - a_j} + \frac{s_j^2}{(\lambda_2 - a_j)^2} \right)}$$

as the Hamiltonian function,

$$\dot{L}(u) = -\mathrm{i}\{\mu_2, L(u)\}. \tag{7.40}$$

This means that the two-point map discretizes a one-parameter family of flows. Having chosen the parameter λ_2 to be equal to any of the poles of the Lax matrix (parameters of the model) a_j, $j = 1, \ldots, n$, the map leads to n different maps, each discretizing the flow with the corresponding Hamiltonian H_j, $j = 1, \ldots, n$. Indeed, taking the limit $\lambda_2 \to a_j$,

$$\lambda_2 = a_j + \varepsilon \qquad \varepsilon \to 0. \tag{7.41}$$

Then we have

$$\mu_2 = \frac{s_j}{\varepsilon} + \frac{H_j}{2s_j} + \mathrm{O}(\varepsilon) \tag{7.42}$$

and in this limit the Lax equation (7.39), (7.40) turns into

$$\dot{L}(u) = -\frac{\mathrm{i}}{2s_j}\{H_j, L(u)\} = \left[L(u), \frac{1}{2s_j(u - a_j)} \begin{pmatrix} s_j^3 & s_j^- \\ s_j^+ & -s_j^3 \end{pmatrix} \right]. \tag{7.43}$$

Let us denote a collection of these maps by $\{B_{P_1}^{H_j}\}_{j=1}^n$. The map $B_{P_1}^{H_k}$ discretizes the flow governed by the Hamiltonian H_k with $\lambda_1 - a_k$ playing the role of the discrete time-step parameter. The map (and its inverse) is defined by the two-point matrix $M(u)$ (7.22) with the following expressions for the variables x and X:

$$x = \frac{A(\lambda_1) - \mu_1}{B(\lambda_1)} = \frac{\tilde{\tilde{s}}_k^3 - s_k}{\tilde{\tilde{s}}_k^-} \tag{7.44}$$

$$X = \frac{(a_k - \lambda_1)B(\lambda_1)s_k^-}{B(\lambda_1)(s_k^3 + s_k) - s_k^-(A(\lambda_1) - \mu_1)} = \frac{(a_k - \lambda_1)\tilde{\tilde{B}}(\lambda_1)\tilde{\tilde{s}}_k^-}{\tilde{\tilde{s}}_k^-(\tilde{\tilde{A}}(\lambda_1) + \mu_1) - \tilde{\tilde{B}}(\lambda_1)(\tilde{\tilde{s}}_k^3 - s_k)}. \tag{7.45}$$

All these maps are explicit Poisson maps, preserving Hamiltonians and having the spectrality property with respect to the pair of variables (λ_1, μ_1).

8. Concluding remarks

One of the very important branches of the theory of finite-dimensional integrable systems is the area of discrete-time integrable systems. Interest in this area was revived at the beginning of the 1990s by Veselov in a series of works (see [3]). He defined integrable Lagrange correspondences as discrete-time analogues of integrable continuous flows, clarified their geometric meaning as finite shifts on Jacobians and gave several important examples. Since then the subject has been developed further by many authors. It is not our intention to review the many important recent contributions made to the field, which would require much more space. Instead, here we only mention briefly the main features of a new recent approach to constructing integrable maps that was introduced in [8, 10, 12], developed in [9] and is also used in this paper, which we refer to as BTs for finite-dimensional integrable systems.

One of the new features of this approach to discrete-time integrability is *the spectrality property* which is a projection on the classical case of the well known (quantum) Baxter equation. It was discovered on the examples of Toda lattice and elliptic Ruijsenaars system in [8] and was generalized to the integrable case of the DST model in [10]. Later, it was observed that the property is universal and that, in effect, it gives a canonical way of parametrizing the corresponding shift on the Jacobian which is characterized by adding a point (λ, μ) to a divisor of points on the spectral curve Γ [9].

A direct consequence of the spectrality property is the *explicitness* of the constructed maps. This new feature, which is an obvious advantage because explicit iterative maps are much more useful than implicit maps (given as a system of nonlinear equations), was clearly demonstrated in [9]. This new approach of constructing *explicitly given maps* has also been adopted and illustrated in detail in this paper.

Several examples of explicit maps were known earlier, such as McMillan's map, but all those cases were exceptional, for in the generic situation, according to Veselov's approach, integrable Lagrange/Poisson correspondences are multi-valued maps, i.e. *correspondences* rather than maps. Using the spectrality property as extra data allows one to overcome this drawback and to construct discrete-time integrable flows as genuine maps.

Another new feature of the proposed construction of integrable time discretizations is an identification of the most elementary, one-point, basic map and construction of composite maps, like the two-point map, as compositions of the one-point map and its inverse. The choice of the matrix $M(u)$ (3.8) generating the one-point map is dictated by the algebraic considerations explained in [12]. In brief, the matrix $M(u)$ should be a simple L-operator of the quadratic algebra,

$$\{L_1(u), L_2(v)\} = [r(u-v), L_1(u)L_2(v)] \tag{8.1}$$

with the same rational r-matrix (2.7) as in the linear algebra (2.6). The number of zeros of the det $M(u)$ is the number of essential Bäcklund parameters, so that the matrix $M(u)$ in (3.8) is one-point and the matrix $M(u)$ in (7.22) is two-point. The fact that the right ansatz for the matrix $M(u)$ obeys the algebra (8.1) usually guarantees that the resulting map will be Poisson: see [12] for details.

In this paper we have observed a new 'spectrality' property of the basic one-point map with respect to the parameter γ in

$$\det M(u) = \gamma(u-\lambda). \tag{8.2}$$

We have also shown that the two-point map factorizes to two one-point maps.

The two-point map constructed above is probably the most general map for the considered $sl(2)$ Gaudin model, meaning that it gives a discretization of continuous flows given by any Hamiltonian H_j, $j = 1, \ldots, n$, from the spectral curve,

$$v^2 = A^2(u) + B(u)C(u) = \alpha^2 + \sum_{j=1}^{n} \left(\frac{H_j}{u-a_j} + \frac{s_j^2}{(u-a_j)^2} \right). \tag{8.3}$$

So, at least in principle, any other integrable map for this model should be a function of the n maps constructed in this paper.

There is no established name for integrable maps with all the qualities mentioned above, namely (i) spectrality, (ii) explicitness, (iii) Poissonicity, (iv) limits to continuous flows and (v) preservation of the same integrals as for the continuous flows which these maps discretize. We have used the same name, BTs, as was used in [8–10, 12].

The application of the constructed maps as exact numerical integrators of the continuous flows is considered in [13].

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